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EE 364 - Spring 2026

Week 2
Supplemental Notes

Rules of Inferance

(the logical schema)

Modus Ponens

(Modus ponendo ponens)

"mode that affirms by affirming"

Premise 1: $P \rightarrow Q$

Premise 2: P

Conclusion: Q

Theorem: $[(P \rightarrow Q) \,\&\, P] \rightarrow Q$

Prf: By truth table

P	Q	$(P$	$\rightarrow$	$Q)$	$\wedge$	P	$\rightarrow$	Q
1	1	1	1	1	1	1	1	1
0	1	0	1	1	0	0	1	1
1	0	1	0	0	0	1	1	0
0	0	0	1	0	0	0	1	0

Modus Tollens

(modus tollendo tollens)

"mode that denies by denying"

Premise 1: $P \rightarrow Q$

Premise 2: $\sim Q$

Conclusion: $\sim P$

Defn: A fallacy is a logical error
(not factual error)

Theorem: $[(P \rightarrow Q) \ \& \ \sim Q] \rightarrow \sim P$

truth → factual truth "grass is green"

truth → logical truth "green is green" ← tautology

Fallacy of affirming the consequent

P1: $P \rightarrow Q$

P2: Q

C: $\therefore P.$

Fallacy of denying the consequent

P1: $P \rightarrow Q$

P2: $\sim P$

C: $\therefore \sim Q$

Conditional: $P \longrightarrow Q$

Contra positive: $\sim Q \longrightarrow \sim P$ ⟵ same logical truth as conditional.

Converse: $Q \longrightarrow P$

i.e. $[P \rightarrow Q] \longleftrightarrow [\sim Q \rightarrow \sim P]$.

Theorem: "Material implication"

$$P \longrightarrow Q = \sim P \vee Q$$

Defn: An argument is <u>VALID</u> iff the premises logically imply the conclusion

(can't make if-part TRUE and then-part FALSE)

A proposition is <u>valid</u> by <u>structure, not content</u>

Defn: An argument is <u>SOUND</u> iff it is valid and all premises are true

Ex: $[(P \vee Q) \to R] \leftrightarrow [P \to (Q \to R)]$.

P	Q	R	$[(P$	$\vee$	$Q)$	$\to$	$R]$	$\leftrightarrow$	$[P$	$\to$	$(Q$	$\to$	$R)]$
1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	1	1	1	1	1	0	1	1	1	1
1	0	1	1	1	0	1	1	1	1	1	0	1	1
0	0	1	0	0	0	1	1	1	0	1	0	1	1
1	1	0	1	1	1	0	0	1	1	0	1	0	0
0	1	0	0	1	1	0	0	0	0	1	1	0	0
1	0	0	1	1	0	0	0	0	1	1	0	1	0
0	0	0	0	0	0	1	0	1	0	1	0	1	0
			1	2	1	3	1	4	1	3	1	2	1

∴ False (since the "final" column is not all 1's)

Defn: Suppose $(\Omega, \mathcal{A}, P)$ is a probability space. Events A and B are (statistically) independent w.r.t. P iff $P(A \cap B) = P(A) \cdot P(B)$

"joint factors into marginals"

Defn: Events $A_1, A_2, \dots$ are independent $\longleftrightarrow$ every finite subset is independent.

Q: Can A be independent of itself?

A: iff $P(\underbrace{A \cap A}_{A}) = P(A) \cdot P(A) \longleftrightarrow P(A) = P(A)^2$

$\therefore$ only if $P[A] = 0$ or $P[A] = 1$.

Q: Can A be independent of X?

A: iff $P(\underbrace{A \cap X}_{A}) = P(A) \cdot \underbrace{P(X)}_{=1} \text{ (CAT)} \longleftrightarrow P(A) = P(A)$

$\therefore$ every set is independent of X.

Q: Does A indep. $B \longrightarrow A \cap B = \emptyset$? (they are disjoint).

A: iff $P(\underbrace{A \cap B}_{\emptyset \text{ by assump.}}) = P(A) \cdot P(B) \longleftrightarrow P[A] = 0$ or $P[B] = 0$.

no in general

Ex: Coin flip w/ $P[\text{head}] = p$ $\therefore P[\text{tail}] = 1-p$.

(independent flips)

Flip twice:

$$P[H_1 \cap H_2] = P[H_1]\cdot P[H_2] = P[H]^2 = p^2$$
$$P[H_1 \cap T_2] = P[H_1]\cdot P[T_2] = P[H]\cdot P[T] = p(1-p)$$
$$P[T_1 \cap H_2] = P[T_1]\cdot P[H_2] = P[T]\cdot P[H] = (1-p)p$$
$$P[T_1 \cap T_2] = P[T_1]\cdot P[T_2] = P[T]^2 = (1-p)^2$$

Note: $P[\{H_1 \cap H_2\} \cup \{H_1 \cap T_2\} \cup \{T_1 \cap H_2\} \cup \{T_2 \cap H_2\}]$

$$= p^2 + 2p(1-p) + (1-p)^2$$

$$= P[\Omega] = 1 \qquad \therefore \; = 1.$$

Q: $P[1 \text{ tail AND } 1 \text{ head (any order)}]$

$$= P[\{H_1 \cap T_2\} \cup \{T_1 \cap H_2\}] \overset{\substack{\text{disjoint} \\ \text{CA}}}{=} P[H_1 \cap T_2] + P[T_1 \cap H_2]$$
$$= 2p(1-p)$$

Q: $P[\text{at least } 1 \text{ tail}] = P[\{H_1 \cap T_2\} \cup \{T_1 \cap H_2\} \cup \{T_1 \cup T_2\}]$.

$$= 2p(1-p) + (1-p)^2$$

$$= 1 - P[(\text{at least } 1 \text{ tail})^c]$$
$$= 1 - P[\text{no head}]$$
$$= 1 - p^2$$

Defn: Conditional Probability $P(\underbrace{B|A}_{\text{conditional}}) = \dfrac{P(\overbrace{A \cap B}^{\text{joint}})}{P(\underbrace{A}_{\text{marginal}})}$ if $P(A) > 0$.

$\therefore$ A & B independent $\longleftrightarrow P(B|A) = P(B)$

$\longleftrightarrow P(A|B) = P(A)$

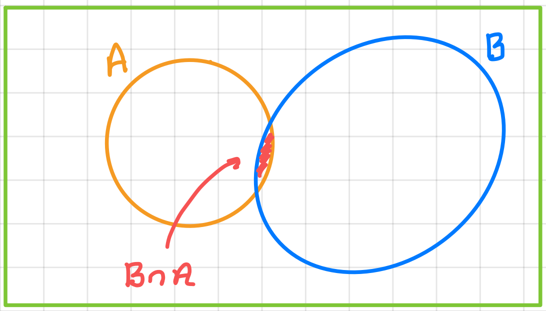
Prf: $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A).$

Important: $P[A] = 0 \not\longrightarrow A = \phi$ $\quad (A = \phi \longrightarrow P[A] = 0)$

Note: Conditional probability is not transitive

$$P[A|B] \neq P[B|A]$$

Intuition: If you know A occured then you constrain the size of sample space of B $\to B \cap A$, $\to B^c \cap A$.



the value $P[A]$ normalizes area to 1.

Ex: Urn experiment w/ 4-numbered balls: #1 and #2 blue
#3 and #4 red

Events
A : blue ball
B : even ball
C : # ball > 2

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{1/4}{1/2} = \frac{1}{2} \quad (\text{note } = P[A])$$

"blue given even"

$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{0}{1/2} = 0 \quad (\text{note } \neq P[A])$$

"blue given # > 2"

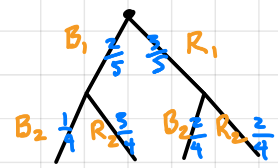
$\therefore$ conditioning can (but may not) affect computed probability.

Note: $P[A \cap B] = P[A \mid B] \cdot P[B]$

$$= P[B \mid A] \cdot P[A]$$

Ex: Urn experiment w/ 5-balls, 2 blue, 3 red

Draw 2 balls: $P[\text{2 blue balls}]$



Consider as 2 "sub" experiments

$$P[B_1 \cap B_2] = P[B_1] \cdot P[B_2 \mid B_1]$$

$$= \frac{2}{5} \cdot \frac{1}{4}$$

$$= \frac{1}{10}$$

Thrm: Suppose A and B are independent then: $P[A \mid B] = P[A]$

$P[B \mid A] = P[B]$

Prf: $P[B \mid A] = \dfrac{P[A \cap B]}{P[A]} = \dfrac{P[B] \cdot \cancel{P[A]}}{\cancel{P[A]}} = P[B]$

and similarly for $P[A \mid B]$.

★ Conditioning does not affect the probability of independent events.

Thrm: Probabilistic Modus Ponens

$$\begin{array}{ll} \text{Premise 1:} & P(B|A) \geq c \\ \text{Premise 2:} & P(A) \geq a \\ \hline \text{Conclusion:} & P(B) \geq a \cdot c \end{array}$$

Pf:

$$\begin{aligned} P(B) &\geq P(A \cap B) && \text{monotonocity} \\ &= P(A) \cdot \frac{P(A\cap B)}{P(A)} && P(A) > 0 \\ &= P(A) \cdot P(B|A) && \text{defn cond. prob.} \\ &\geq a \cdot c && \text{premise 1 and 2.} \end{aligned}$$

QED

Check: $a = c = 1 \quad \therefore P(B) = 1$

Thrm: Probabilistic Modus Tollens

$$\begin{array}{ll} \text{Premise 1:} & P(B|A) \geq c > 0 \\ \text{Premise 2:} & P(B) \leq b. \\ \hline \text{Conclusion:} & P(A) \leq \min\left(1, \frac{b}{c}\right) \end{array}$$

Pf:

$$\begin{aligned} P(A) &\leq P(A)\left(\frac{P(B)}{P(A\cap B)}\right) && \text{since } P(A\cap B) \leq P(B) \\ &= \frac{P(B)}{P(B|A)} \\ &\leq \frac{b}{P(B|A)} && \text{premise 1.} \\ &\leq \frac{b}{c} && \text{premise 2} \end{aligned}$$

Always $P(A) \leq 1 \quad \therefore P(A) \leq \min\left(1, \frac{b}{c}\right)$

QED.

Proof by induction. Prove $A(n)$ holds $\forall n$.

(1) Prove $A(1)$

(2) Show $A(n) \longrightarrow A(n+1)$

Ex: $n^2 + n$ even $\forall n$

- Basis, $n = 1$ $\quad 1^2 + 1 = 2 = 2 \cdot k$ (for $k=1$) $\quad \therefore$ even $\quad$ QED basis
- Induction, Suppose $n^2 + n = 2 \cdot k'$ for some $k' \in \mathbb{Z}^+$.

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = \underbrace{n^2 + n}_{k'} + 2\underbrace{(n+1)}_{k''} = 2(k' + k'')$$

QED.

Theorem: ("Booles inequality") $\quad P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} P(A_k)$

Prf: (by induction on $n = 2, 3, 4, \ldots$)

Basis Step: $\quad n = 2.$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \qquad \text{Add. theorm}$$

$$\leq P(A_1) + P(A_2) = \sum_{k=1}^{2} P(A_k) \qquad P[A] \geq 0$$

QED-basis

Induction Step: - Assume holds for n.

- Derive that it holds for $n+1$

Induction Hypothesis (IH).

$$P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} P(A_k)$$

$$\Longrightarrow \quad P\left(\bigcup_{k=1}^{n+1} A_k\right) = P\left(\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right)$$

$$\leq P\left(\bigcup_{k=1}^{n} A_k\right) + P(A_{n+1})$$

$$\leq \sum_{k=1}^{n} P(A_k) + P(A_{n+1})$$

$$= \sum_{k=1}^{n+1} P(A_k) \qquad \text{QED}$$

Thrm: ("Inclusion-Exclusion") Note: generalizes CA.

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) - \sum_{i<j}\sum P(A_i \cap A_j) + \sum_{i<j<k}\sum\sum P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Technique: Unions are usually hard to work with, ∴ so either

1) De Morgan's
2) Additivity (find disjoint)

Theorem: ("Multiplication theorem")

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1})$$

no dependence.

Compare to: Independence: $= P(A_1) \cdot P(A_2) \cdots P(A_n) = \prod_{k=1}^{n} P(A_k)$

Markov: $= P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_2) \cdots P(A_n \mid A_{n-1})$

depend only on previous 1.

$$= P(A_1) \cdot \prod_{k=2}^{n} P(A_k \mid A_{k-1})$$

Ex: Drawing 3 aces

w/ replacement (independent samples) $P[A_1 \cap A_2] = P[A_1] \cdot P[A_2]$

w/o replacement $P[A_1 \cap A_2] = P[A_1] \cdot P[A_2 \mid A_1]$

$P[\text{draw 3 aces}]$ w/o replacement $= P[A_1 \cap A_2 \cap A_3]$

$$= P[A_1] \cdot P[A_2 \mid A_1] \cdot P[A_3 \mid A_1 \cap A_2]$$

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50}$$

$$= \frac{1}{5525} \quad (\approx 0.00018)$$

Pf: (by induction on $n = 2, 3, 4, \ldots$)

Basis Step: $n = 2$.

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1)$$ by defn cond. prob.

QED-basis

Induction Step: – Assume holds for n.

– Derive that it holds for $n+1$

Induction Hypothesis (IH):

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1})$$

$$\Rightarrow P\left(\bigcap_{k=1}^{n+1} A_k\right) = P\left(\bigcap_{k=1}^{n} A_k \cap A_{n+1}\right)$$

$$\stackrel{\text{assoc.}}{=} P\left(\left(\bigcap_{k=1}^{n} A_k\right) \cap A_{n+1}\right)$$

$$= P\left(\bigcap_{k=1}^{n} A_k\right) \cdot P\left(A_{n+1} \,\middle|\, \bigcap_{k=1}^{n} A_k\right)$$

$$= P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1}) \cdot P\left(A_{n+1} \,\middle|\, \bigcap_{k=1}^{n} A_k\right)$$

QED.

Defn: Suppose $(\Omega, \mathcal{Q})$ is a measureable space. Then $\{H_k\} \subset \mathcal{Q}$ partition Ω ("is a partition") iff

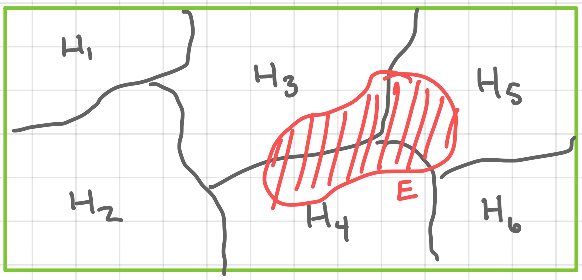
union that does not overlap $\therefore$ CA

(1) $\bigcup_k H_k = \Omega$

(2) pairwise disjont $H_i \cap H_j = \phi$ if $i \neq j$

Theorem: (of Total Probability)

$$P(E) = \sum_k P(H_k) \cdot P(E|H_k) \quad \text{if } \{H_k\} \text{ partition } \Omega$$



Hypotheses: H_k

Evidence: E

Prf: $P(E) = P[E \cap \Omega]$

$= P[E \cap (\bigcup_k H_k)]$ since $\{H_k\}$ partiton Ω

$= P(\bigcup_k (E \cap H_k))$

$= \sum_k P(E \cap H_k)$ since $\{H_k\}$ partition Ω

$= \sum_k P(H_k) \cdot P(E|H_k)$ defn of cond. prob. $P(B|A) = \frac{P(A \cap B)}{P(A)}$

QED

Ex: Urn w/ 5-balls, 2 blue, 3 red

P[second ball is red]

events: $B_1 = \{b_1 \cap r_2, b_1 \cap b_2\}$
$\sim B_1 = R_1 = \{r_1 \cap r_2, r_1 \cap r_2\}$

partition: B_1 vs. $\sim B_1$

$$\therefore P[r_2] = P[r_2 | b_1] \cdot P[b_1] + P[r_2 | r_1] \cdot P[r_1]$$
$$= 3/4 \cdot 2/5 + 1/2 \cdot 3/5$$
$$= 3/5 \quad (60\%)$$

Note: (another representation)

Data = Evidence

Hypotheses $H_1, H_2, \ldots, H_n$ partition sample space

$$P[E] = \sum_{k=1}^{n} \underbrace{P[E | H_k]}_{\text{"likelihood"}} \cdot \underbrace{P[H_k]}_{\text{"prior"}}$$

Later in this course:

Total Expectation $\quad E_X[X] = E_Y[E[X|Y]]$

Total Variance $\quad V_X[X] = E_Y[V[X|Y]] + V_Y[E[X|Y]]$.

extremely important for problem solving

Special case: $\Omega = A \cup A^c$

$$\therefore P(B) \overset{\text{total}}{\underset{\text{prob.}}{=}} P(A)\cdot P(B|A) + P(A^c)\cdot P(B|A^c)$$

Note: for <u>any</u> B.

<u>Thrm</u>: $P[B|A] = \dfrac{P[A|B]\cdot P[B]}{P[A]}$ for any event B. (if $P[A]>0$)

<u>Prf</u>: $P[B|A] = \dfrac{P[A\cap B]}{P[A]} \cdot \dfrac{P[B]}{P[B]} = \dfrac{P[A|B]\cdot P[B]}{P[A]}$ QED.

<u>Theorem</u>: (Baye's Theorem)

$$P(H_j | E) = \frac{P(E|H_j)\cdot P(H_j)}{\sum_k P(E|H_k)\cdot P(H_k)}$$ if $\{H_k\}$ partitions Ω.

<u>Prf</u>: $P(H_j|E) = \dfrac{P(E\cap H_j)}{P(E)}$ defn of cond. prob

$= \dfrac{P(H_j)\cdot P(E|H_j)}{P(E)}$ " " "

$= \dfrac{P(H_j)\cdot P(E|H_j)}{\sum_k P(H_k)\cdot P(E|H_k)}$ total prob.

<u>QED</u>

$P(H_j)$: <u>Prior</u>

$P(H_j|E)$: <u>Posterior</u>

$P(E|H_j)$: <u>Likelihood</u>

$\therefore P(H|E) = \dfrac{a}{a+b}$

$P[H^c|E] = \dfrac{b}{a+b}$

IMAC format

I: Issue

- associative memory ("spot the issue")

M: Math rule

- rote memorization

A: Apply math to facts (Analysis)

- pattern matching
- element because fact.

C: Conclusion

- deductive logic

Mnemonic:

"Party Unconditionally To Conquer Bayes"

P : Partition $\{H_k\}$ or $A \cup A^c$

U : Unconditional probability $P(B)$

T : Total Probability

C : Conditional Probability $P(B|A)$

B : Bayes Theorem

Issue Spotting Sequence

- Is there a partition ?
  - Unconditional probability $P(B)$?

    $\therefore$ Use total probability

  - Conditional probability $P(B|A)$?

    $\therefore$ Use Baye's Theorem

Ex: Special case H vs H^c

$H \cup H^c = X$
$H \cap H^c = \phi$ $\therefore$ H and H^c partition Ω

$$P[H \mid E] = \frac{P[E \mid H] \cdot P[H]}{P[E \mid H] \cdot P[H] + P[E \mid H^c] \cdot P[H^c]}$$

hypothesis evidence

Ex: Cancer (H) vs. not Cancer (H^c)

Evidence = Test w/ diagnostic:

$\therefore$ prior → 1% of women have breast cancer $\therefore$ 99% do not

test
- 80% detect cancer when it is there — 20% false neg.
- 9.6% detect cancer when it is not there — 90.4% true neg.

Test	Cancer (1%) H	~Cancer (99%) H^c
+	TP 0.80	FP 0.096
−	FN 0.20	TN 0.904
	$\Sigma = 1$	$\Sigma = 1$

$$P[\text{Cancer} \mid +] = \frac{P[+ \mid \text{Cancer}] \cdot P[\text{Cancer}]}{P[+ \mid \text{Cancer}] \cdot P[\text{Cancer}] + P[+ \mid \sim\text{Cancer}]\, P[\sim\text{Cancer}]}$$

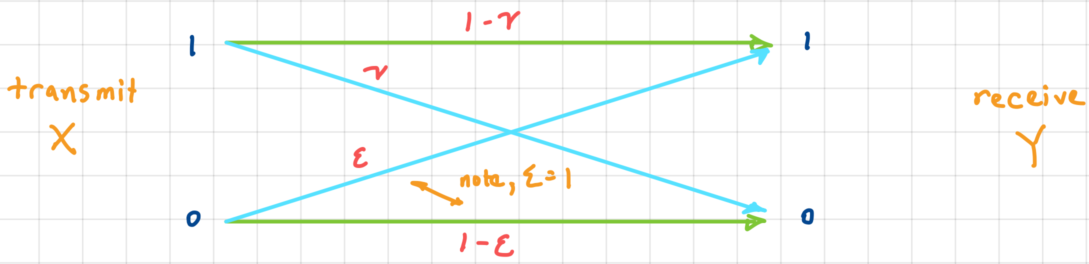
$$= \frac{(0.80)(0.01)}{(0.80)(0.01) + (0.96)(0.99)}$$

$$= 0.0776 \quad (\sim 7.76\%)$$

Note: $\therefore$ get second opinion.

Ex: Binary Channel

Problem: Transmit a single noisy bit. Guess transmitted given received.



$P[Y=1 \mid X=1] = 1-\gamma$ $\qquad$ $P[Y=0 \mid X=1] = \gamma$

$P[Y=1 \mid X=0] = \varepsilon$ $\qquad$ error $\qquad$ $P[Y=0 \mid X=0] = 1-\varepsilon$

Defn: A binary symmetric channel has $\gamma = \varepsilon$.

Q: Suppose $Y=1$ (received 1). Guess $X=0$ or $X=1$?

A: It depends on γ and ε.

Choose hypothesis ($X=0$ or 1) that has higher probability, i.e.

estimate $\hat{X} = 1$ if $P[X=1 \mid Y=1] > P[X=0 \mid Y=1]$ $\qquad$ "pick biggest"

$\hat{X} = 0$ else

Defn: The MAP (maximum a priori) estimate is the hypothesis that maximizes the probability given the observation: $\hat{H}^{MAP} = \arg\max_k P(H_k \mid E)$

In our problem: $P[X=1 \mid Y=1] \gtrless P[X=0 \mid Y=1]$

iff

Bayes: $$\frac{P[Y=1 \mid X=1]\cdot P[X=1]}{\cancel{P[Y=1 \mid X=1]\cdot P[X=1] + P[Y=1 \mid X=0]\cdot P[X=0]}} \gtrless \frac{P[Y=1 \mid X=0]\cdot P[X=0]}{\cancel{P[Y=1 \mid X=1]\cdot P[X=1] + P[Y=1 \mid X=0]\cdot P[X=0]}}$$

$$\underbrace{P[Y=1 \mid X=1]}_{1-\gamma} \cdot \underbrace{P[X=1]}_{p} \underset{<}{>} \underbrace{P[Y=1 \mid X=0]}_{\varepsilon} \cdot \underbrace{P[X=0]}_{1-p}$$

odds

iff $$\frac{P[X=1]}{P[X=0]} \underset{<}{>} \frac{\varepsilon}{1-\gamma}$$

Suppose $P[X=1] = P[X=0]$ $\Longrightarrow$ $\frac{0.50}{0.50} \underset{<}{>} \frac{\varepsilon}{1-\gamma}$

transmit 50% 1 and 50% 0

$\therefore$ if $Y=1$, choose $\hat{X}=1$ if $\frac{\varepsilon}{1-\gamma} < 1$.

$\hat{X}=0$ if $\frac{\varepsilon}{1-\gamma} > 1$

MAP hypothesis test.

$P[\text{flip}] < P[\text{not flip}]$

$\varepsilon < 1-\gamma$

$\varepsilon > 1-\gamma$

$P[\text{flip}] > P[\text{not flip}]$